Problem Statement:

Let $f : \mathbb{R} \to (\infty, 2]$ be the function defined by $f(x) = 2 - x^2$ for each $x \in \mathbb{R}$. Show that f is surjective.

Proof:

To show that f is surjective, we need to show that it satisfies the property

$$\big(\forall y \in (-\infty, 2]\big)\ (\exists x \in \mathbb{R})\ [f(x) = y]$$

Suppose y is an arbitrary real number not greater than 2. Then, it remains to be shown that there exists some real x such that $f(x) = y$.

Find an x: Let $f(x) = y$. Then solving for x, we obtain

$$\begin{aligned} y &= 2 - x^2 \\ x^2 &= 2 - y \\ x &= \sqrt{2 - y} \end{aligned}$$

Thus, choose $x = \sqrt{2 - y}$.

Verify: Next we must verify that this choice of x satisfies the domain and propositional function.

Domain: Since $y \leq 2$, then $2 - y \geq 0$. This implies that $x = \sqrt{2 - y} \in \mathbb{R}$ as required. (Strictly speaking, it will be nonnegative; but that is irrelevant to this problem.)

Proposition: The choice of x must satisfy the propositional function $f(x) = y$. Substitution yields

$$\begin{aligned} f(x) &= f\left(\sqrt{2 - y}\right) \\ &= 2 - \left(\sqrt{2 - y}\right)^2 \\ &= 2 - (2 - y) \\ &= 2 - 2 + y \\ &= y \end{aligned}$$

Thus, $f(x) = y$ as required.

This proves the existential statement to be true. Furthermore, since our choice of y was arbitrary, the original universal statement is true. This shows that the required property is satisfied; and this, in turn, implies that the function f is surjective, as was to be proved. □